

# Spinor Formulations for Gravitational Energy-Momentum

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**ABSTRACT** We first describe a class of spinor-curvature identities (SCI) which have gravitational applications. Then we sketch the topic of gravitational energy-momentum, its connection with Hamiltonian boundary terms and the issues of positivity and (quasi)localization. Using certain SCIs several spinor expressions for the Hamiltonian have been constructed. One SCI leads to the celebrated Witten positive energy proof and the Dougan-Mason quasilocalization. We found two other SCIs which give alternate positive energy proofs and quasilocalizations. In each case the spinor field has a different role. These neat expressions for gravitational energy-momentum have much appeal. However it seems that such spinor formulations just have no room for angular momentum; which leads us to doubt that spinor formulations can really correctly capture the elusive gravitational energy-momentum.

**Keywords:** Gravitation, energy-momentum, positive energy, quasi-local quantity, Hamiltonian.

## 1 Introduction

One of the most outstanding results in classical gravitation theory (more specifically we mean in GR: general relativity, Einstein's gravity theory), obtained via Clifford algebra and spinor methods, was Witten's positive energy proof [34]. This seminal work (which was inspired by an analogous result in quantum supergravity [10, 14]) led to many new ideas regarding gravitational energy and its localization. To appreciate this work and its importance we need to recall some facts about one of nature's most elusive quantities: gravitational energy.

A suitable expression which could provide a physically reasonable description of the energy-momentum density for gravitating systems has long been sought. All candidates had several shortcomings. In particular they

violated a fundamental theoretical requirement—that gravitational energy should be positive—as well as requirements concerning localization and reference frame independence. Since Witten’s positive energy proof (unlike the earlier indirect proof of Schoen and Yau [30]) can be understood in terms of the Hamiltonian, the Hamiltonian density associated with this proof provides a *locally positive localization*—and thus has real promise as a *truly physical* energy-momentum density for gravitational fields. To fulfill this promise certain features need further consideration; an outstanding one concerns the role of—and even the need for—the spinor field, which seemed rather mysterious.

Here we give a survey (further details can be found in the references) of three (classical, commuting, non-supersymmetric) spinor/Clifford algebra formulations of the GR Hamiltonian known to us. We examine the underlying mathematics (spinor-curvature identities), outline their associated positive energy proofs and energy-momentum quasilocalizations, and note the various distinct roles of the spinor field. Although they give good expressions for energy-momentum, these spinor expressions do not seem to have to the proper qualities for giving a good description of relativistic angular momentum. Hence they apparently do not succeed in giving a full physical description for the energy-momentum density of asymptotically flat gravitating systems.

The plan of this work is as follows. In section 2 we present our notation and conventions for geometric objects, forms, the Dirac algebra, spinors, and “Clifforms”. In section 3 a succinct presentation of a class of *spinor curvature identities* is given; three special cases which have gravitational applications are noted. Then in section 4 the topic of gravitational energy-momentum is discussed; we note the fundamental theoretical requirement of positivity (finally proved over 20 years ago) and the still outstanding issue of the localization (or quasi-localization) of gravitational energy-momentum. In section 5 we explain the Hamiltonian approach to energy-momentum, noting the important roles played by the Hamiltonian boundary term; the standard “ADM” Hamiltonian (albeit in non-standard variables) for GR along with a good choice of boundary term (which necessarily requires reference values) are presented. In section 6 we consider three alternate GR Hamiltonians obtained via certain spinor curvature identities along with suitable associated spinor field reparameterizations. The first leads to the celebrated Witten positive energy proof; it also gives the famous Dougan-Mason energy-momentum quasilocalization. The second alternate Hamiltonian uses  $SU(2)$  spinors (i.e., the spinors of 3-dimensional space) in a way similar to, but distinct from, the first case; it yields another proof and quasilocalization. The third alternative is fundamentally quite different; unlike the former cases (where the spinor field was introduced as an alternate reparameterization of the Hamiltonian) now the spinor field enters into the Lagrangian as a dynamic physical field; this case yields yet another positivity proof and quasilocalization. The following section 7 notes that

(i) the spinor formulations have the extremely nice property of not requiring any explicit reference values, (ii) in all cases the *Hamiltonian boundary variation principle* reveals the associated boundary conditions, and (iii) the various distinct roles played by the spinor fields in these three formulations. In section 8 consideration is given to the other conserved quantities of an asymptotically flat space: angular momentum and the center-of-mass moment. The key role played by a certain type of term in the conventional Hamiltonian and the absence of this type of term in all of the aforementioned spinor formulations is noted. The concluding section summarizes the virtues of the spinor Hamiltonian expressions; we note, however, that the limitation in connection with angular momentum and especially the center-of-mass moment raises grave doubts as to whether these spinor formulations are really properly representing the physics of gravitating systems.

## 2 Conventions

Geometry (especially with a metric compatible connection, as is assumed here) can be conveniently described using differential forms. The basis one-forms  $\vartheta^\mu := e^\mu_i dx^i$  (dual to the basis vectors  $e_\nu$ ) are chosen to be orthonormal (and are generally non-holonomic). The metric is then given by  $g = g_{\mu\nu} \vartheta^\mu \otimes \vartheta^\nu$  with  $g_{\mu\nu}$  constant. (We will not need the coordinate frame metric components  $g_{ij} = g_{\mu\nu} e^\mu_i e^\nu_j$ .) The connection is described by the one-form  $\Gamma^{\mu\nu} = \Gamma^{\mu\nu}_i dx^i$ . Because of the metric compatibility condition (and the the frame type choice) the connection one-form is antisymmetric:  $\Gamma^{\mu\nu} \equiv \Gamma^{[\mu\nu]}$ . In general the connection may have torsion, which is neatly described by the 2-form field

$$T^\mu := D\vartheta^\mu := d\vartheta^\mu + \Gamma^\mu_\nu \wedge \vartheta^\nu = (1/2)T^\mu_{\alpha\beta} \vartheta^\alpha \wedge \vartheta^\beta. \quad (2.1)$$

The curvature is also described by a 2-form field:

$$R^\alpha_\beta := d\Gamma^\alpha_\beta + \Gamma^\alpha_\gamma \wedge \Gamma^\gamma_\beta = (1/2)R^\alpha_{\beta\mu\nu} \vartheta^\mu \wedge \vartheta^\nu. \quad (2.2)$$

The unit volume element (in 4-dimensional spacetime) is given by  $\eta := \vartheta^0 \wedge \vartheta^1 \wedge \vartheta^2 \wedge \vartheta^3$ . We often use the dual Grassmann basis, which can be constructed from  $\eta$  by contraction (aka the interior product):  $\eta_\mu := i_{e_\mu} \eta$ ,  $\eta_{\mu\nu} := i_{e_\mu} \eta_\nu$ ,  $\eta_{\mu\nu\alpha} := i_{e_\mu} \eta_{\nu\alpha}$ , and  $\eta_{\mu\nu\alpha\beta} := i_{e_\mu} \eta_{\nu\alpha\beta}$ , the latter is the totally antisymmetric Levi-Civita tensor. Alternately these objects can be obtained via the Hodge dual:  $\eta^{\mu\nu\dots} := *(\vartheta^\mu \wedge \vartheta^\nu \wedge \dots)$ . A succinct notation, neatly suited to our material, is Geometric (Clifford) Algebra valued forms, sometimes referred to as *Clifforms* [11, 21, 13]. With the Dirac conventions

$$\gamma_\alpha \gamma_\beta + \gamma_\beta \gamma_\alpha = 2g_{\alpha\beta} = \text{diag}(+1, -1, -1, -1), \quad (2.3)$$

$$\gamma_{\alpha\beta\dots} := \gamma_{[\alpha}\gamma_{\beta}\dots], \quad \gamma := \gamma^0\gamma^1\gamma^2\gamma^3, \quad (2.4)$$

$$\bar{\psi} := \psi^\dagger \beta, \quad \beta \equiv \beta^\dagger, \quad (\beta\gamma^\mu)^\dagger \equiv \beta\gamma^\mu, \quad (2.5)$$

we define the *frame* (a vector valued one-form), the *torsion* (a vector valued 2-form), the *connection* (a bivector valued one-form), and the *curvature* (a bivector valued 2-form), respectively, by

$$\vartheta := \vartheta^\mu \gamma_\mu, \quad T := D\vartheta = d\vartheta + \Gamma \wedge \vartheta + \vartheta \wedge \Gamma = T^\mu \gamma_\mu, \quad (2.6)$$

$$\Gamma := (1/4)\Gamma^{\mu\nu}\gamma_{\mu\nu}, \quad R := d\Gamma + \Gamma \wedge \Gamma = (1/4)R^{\mu\nu}\gamma_{\mu\nu}. \quad (2.7)$$

The differentials of Dirac spinors are

$$D\psi := d\psi + \Gamma\psi, \quad D\bar{\psi} := d\bar{\psi} - \bar{\psi}\Gamma, \quad (2.8)$$

$$D^2\psi \equiv R \wedge \psi, \quad D^2\bar{\psi} \equiv -\bar{\psi} \wedge R. \quad (2.9)$$

### 3 Some spinor curvature identities

A key to our work is certain *spinor-curvature identities* (SCIs) [27]. They readily follow from

$$\begin{aligned} d[\bar{\psi}A \wedge D(B\psi) - (-1)^a D(\bar{\psi}A) \wedge B\psi] \equiv \\ 2D(\bar{\psi}A) \wedge D(B\psi) + (-1)^a \bar{\psi}A \wedge D^2(B\psi) - (-1)^a D^2(\bar{\psi}A) \wedge B\psi. \end{aligned} \quad (3.1)$$

(Here  $A$  and  $B$  are Clifford algebra valued forms of rank  $a$  and  $b$ .) Using  $D^2(B\psi) = R \wedge B\psi$ ,  $D^2(\bar{\psi}A) = -\bar{\psi}A \wedge R$  we find the SCIs

$$\begin{aligned} 2D(\bar{\psi}A) \wedge D(B\psi) \equiv 2(-1)^a \bar{\psi}A \wedge R \wedge (B\psi) \\ + d[\bar{\psi}A \wedge D(B\psi) - (-1)^a D(\bar{\psi}A) \wedge B\psi]. \end{aligned} \quad (3.2)$$

Qualitatively  $(D\psi)^2 \equiv \psi^2 R$  plus a total differential. One can get various linear combinations of the curvature depending on the choice of  $A, B$ . We have found three special cases with gravitational applications; they contain, respectively, (i) the Einstein 3-form in 4 dimensions, (ii) the scalar curvature in 3 dimensions, and (iii) the scalar curvature in 4 dimensions.

We know that one can also have identities of the general form (3.2) with the spinor field  $\psi$  replaced by a vector or tensor. However we have not found any such *tensor-curvature identities* with the property that the term linear in the curvature reduces to the Einstein or scalar curvature—which is what we need for our gravitational applications. This technical point is apparently the reason why we need to use spin 1/2 to help clarify certain things about gravity, a field which is fundamentally spin 2.

## 4 The energy-momentum of gravitating systems

Isolated gravitating systems have gravitational fields which are asymptotically flat (very far away the field is essentially Newtonian). For such spaces the total energy-momentum (EM) is well defined [22]. An essential fundamental theoretical requirement (from thermodynamics and stability: otherwise systems could emit an unlimited amount of energy while decaying deeper into ever more negative energy states) is that the energy of gravitating systems should be *positive*. Essentially this means that gravity acts like a purely attractive force. (The total energy is just  $E = Mc^2$ , with  $M$  being the apparent asymptotic Newtonian mass; thus positive energy means  $M > 0$ , hence an attractive force.) This was finally rigorously proved for GR by Schoen and Yau [30] via an indirect argument. Soon thereafter Witten [34] gave his celebrated direct spinorial positive energy proof (see also [23]).

Although positivity has been settled, the *location* of the energy of gravitating systems has remained an outstanding issue since Einstein's day. Sources (which have a well defined local EM density) exchange EM with the gravitational field —*locally*— hence it was natural to expect a *local* gravitational EM density. But no suitable expression has been found. Standard techniques (e.g., translation symmetry and Noether's theorem) give only non-covariant (coordinate dependent) *pseudotensor* expressions (for recent discussions see [4, 5]). It was eventually realized that this localization problem is entirely consistent with the *equivalence principle*: 'gravity is not observable at a point' [22]. Nowadays the more popular idea is *quasilocal* EM (i.e., associated with a closed 2 surface) [3].

## 5 The Hamiltonian approach

A good definition of energy is: the value of the Hamiltonian. For both a finite or infinite region the Hamiltonian includes a 3-volume term and a bounding 2-surface integral term:

$$H(N) = \int_V N^\mu \mathcal{H}_\mu + \oint_{S=\partial V} \mathcal{B}(N), \quad (5.1)$$

here  $N$  is the spacetime vector field describing the evolution (timelike displacement) of the spatial volume  $V$ .

For gravitating systems it follows from Noether's theorem and local translation (diffeomorphism) symmetry that  $\mathcal{H}_\mu \propto$  field eqns (the initial value constraints). Consequently the volume term (although it serves to generate the Hamiltonian equations) has vanishing numerical value. The

boundary term plays a doubly important role: it gives the value of the quasilocal quantities, and it also gives the boundary conditions. In addition to its dependence on the dynamic fields and the displacement vector field, the boundary term generally also depends on a choice of reference fields (which determines the “zero” for the quasilocal values). There is yet considerable freedom. Indeed, at least formally, one could say that there are an infinite number of possible choices for the boundary term  $\mathcal{B}$ ; each corresponds to a distinct selection among the infinite number of conceivable choices for the boundary conditions. Thus additional criteria are very much needed. We proposed “covariant-symplectic” boundary conditions; it turns out that there are only two choices that satisfy this property (essentially they correspond to Dirichlet and Neumann boundary conditions) [4, 8, 6, 7].

For GR in terms of differential forms the standard “ADM” Hamiltonian is given by the spatial integral of the 3-form

$$\mathcal{H} = -N^\mu R^{\alpha\beta} \wedge \eta_{\alpha\beta\mu} - i_N \Gamma^{\alpha\beta} D\eta_{\alpha\beta} + d\mathcal{B}(N). \quad (5.2)$$

This is easily verified if one just notes that  $R^{\alpha\beta} \wedge \eta_{\alpha\beta\mu} \equiv -2G^\nu{}_\mu \eta_\nu$  is a 3-form version of the Einstein tensor. When (5.2) is integrated over space only the coefficient of  $\eta_0$  contributes; that coefficient is  $2G^0{}_\mu$ , the well known covariant components that make up the ADM Hamiltonian, see, e.g., [22, 18].

The total differential, when integrated over a spatial region, yields an integral over the boundary of the region. The weak field limit (which applies asymptotically) fixes the form of  $\mathcal{B}$ , but only to linear order. Our best choice for the boundary term in general is

$$\mathcal{B}(N) = \Delta\Gamma^{\alpha\beta} \wedge i_N \eta_{\alpha\beta} + \overset{\circ}{D}^{[\beta} \overset{\circ}{N}^{\alpha]} \Delta\eta_{\alpha\beta}, \quad (5.3)$$

where  $\Delta\Gamma := \Gamma - \overset{\circ}{\Gamma}$ ,  $\Delta\eta := \eta - \overset{\circ}{\eta}$ . Here  $\overset{\circ}{\Gamma}$ ,  $\overset{\circ}{\eta}$  indicate reference values—usually taken to be the flat space field values. (Note: all of the quasilocal quantities vanish when the dynamic fields take on the reference values on the boundary.) The boundary term (5.3) yields quasilocal values with good limits asymptotically [15, 16] and has good correspondence with other well established expressions [6].

The significance of the choice of boundary term (5.3) is revealed by the variation of the Hamiltonian:

$$\delta\mathcal{H}(N) = \delta\vartheta^\alpha \wedge \frac{\delta\mathcal{H}(N)}{\delta\vartheta^\alpha} + \delta\Gamma^{\alpha\beta} \wedge \frac{\delta\mathcal{H}(N)}{\delta\Gamma^{\alpha\beta}} + di_N(\Delta\Gamma^{\alpha\beta} \wedge \delta\vartheta^\mu \wedge \eta_{\alpha\beta\mu}). \quad (5.4)$$

In addition to the field equation terms (we do not need their explicit functional form here) we obtain a Hamiltonian boundary variation term. It has a *symplectic structure* [19] which, according to the *boundary variation principle*, reveals which variables are to be held fixed. In this case we should fix (certain projected components of) the orthonormal frame  $\vartheta^\mu$ , (geometrically that is equivalent to holding the metric fixed) [8, 6, 7].

## 6 Spinor expressions

Using certain special cases of the general spinor curvature identity (3.2), we obtain three alternate spinor formulations for the GR Hamiltonian and its boundary term. From each we get both a positive energy proof and a quasilocal EM expression. We discuss the first case in some detail and then briefly survey the novel features in the other two cases.

### 6.1 The Witten spinor approach

In the above ADM Hamiltonian 3-form (5.2), use a spinor parameterization for the Hamiltonian displacement

$$N^\mu = \bar{\psi} \gamma^\mu \psi. \quad (6.1)$$

With an appropriately adjusted boundary term, a suitable *spinor-curvature identity* then gives the Hamiltonian 3-form associated with the famous Witten positive energy proof:

$$\begin{aligned} \mathcal{H}_w(\psi) &:= 4D\bar{\psi} \wedge \gamma \vartheta \wedge D\psi - i_N \Gamma^{\alpha\gamma} D\eta_{\alpha\gamma} \\ &\equiv -N^\mu R^{\alpha\gamma} \wedge \eta_{\alpha\gamma\mu} - i_N \Gamma^{\alpha\gamma} D\eta_{\alpha\gamma} + d\mathcal{B}_w, \end{aligned} \quad (6.2)$$

where

$$\mathcal{B}_w := 2(\bar{\psi} \gamma \vartheta \wedge D\psi + D\bar{\psi} \wedge \gamma \vartheta \psi). \quad (6.3)$$

We stress that this is an acceptable alternate form for (5.2), the GR Hamiltonian [24]. Note that the spinor field can take on almost any value, as long as it is asymptotically constant. For such spinor fields the boundary term (6.3), notwithstanding appearances, actually gives the same asymptotic values as those given by (5.3).

For positive energy, first note that, because of vanishing torsion, one of the Hamiltonian terms vanishes:  $D\eta_{\mu\nu} = T^\lambda \wedge \eta_{\mu\nu\lambda} = 0$ . The Hamiltonian density is thus

$$\mathcal{H}_w(\psi) = 4(D_\alpha \bar{\psi} \gamma \gamma_\lambda D_\beta \psi) \vartheta^\alpha \wedge \vartheta^\lambda \wedge \vartheta^\beta \equiv 4(D_\alpha \bar{\psi} \gamma \gamma_\lambda D_\beta \psi) \eta^{\alpha\lambda\beta\mu} \eta_\mu. \quad (6.4)$$

Now, using the grade 3 identity

$$\gamma_\lambda \gamma_{\mu\nu} + \gamma_{\mu\nu} \gamma_\lambda \equiv 2\gamma_{\lambda\mu\nu} \equiv 2\eta_{\lambda\mu\nu\kappa} \gamma^\kappa, \quad (6.5)$$

the Hamiltonian density becomes

$$\mathcal{H}_w(\psi) = 2D_\alpha \bar{\psi} (\gamma^{\alpha\beta} \gamma^\mu + \gamma^\mu \gamma^{\alpha\beta}) D_\beta \psi \eta_\mu. \quad (6.6)$$

When integrated over space only the  $\mu = 0$  term survives; this means that the other two indices must be spatial. Finally, using  $\gamma^{ab} \equiv \gamma^a \gamma^b - g^{ab}$ ,

$g^{ab} = -\delta^{ab}$  and the usual type of representation wherein  $\beta = \gamma^0$ , the Hamiltonian density has the 3+1 (space+time) decomposition

$$\mathcal{H}_w(\psi) \simeq 2D_a \bar{\psi} (\gamma^{ab} \gamma^0 + \gamma^0 \gamma^{ab}) D_b \psi \eta_0 \propto |D_k \psi|^2 - |\gamma^k D_k \psi|^2. \quad (6.7)$$

Now, exploiting the freedom in the choice of  $\psi$ , note that the Hamiltonian density (and consequently the energy) is manifestly non-negative, for any  $\psi$  solving the (3-dimensional, elliptic) *Witten equation*:  $\gamma^k D_k \psi = 0$ . (This proof, due to Witten, was later derived directly from the classical, *anti-commuting* spinor, supergravity result [17, 9]; there is an interesting argument that GR has positive energy *because* it admits a supersymmetric extension.)

Moreover we get a ‘locally positive localization’ for gravitational energy (albeit the *localization* actually depends on the solution of an elliptic equation and hence really depends on the fields globally). Examined in more detail, the argument also shows, (i) that the 4 energy-momentum  $P^\mu$  is future time-like, and (ii)  $P^\mu$  vanishes only for Minkowski space.

Altogether these are very beautiful arguments for some nice important results. Nevertheless, more than 20 years later, it is still not clear as to how much the Witten argument really captures the correct physics.

Consider the (static, spherically symmetric) Schwarzschild solution in an isotropic Cartesian frame ( $\vartheta^0 = N dt$ ,  $\vartheta^i = \varphi^2 dx^i$ , with  $\varphi = 1 + m/2r$ ,  $N\varphi = 1 - 2m/r$ ). The Witten equation is easily solved:  $\psi = \varphi^{-2} \psi_{const}$ . One can then substitute this solution into the Hamiltonian and the boundary term, and thereby conclude that, for the Schwarzschild solution, 1/8 of the energy has been “localized” within the black hole horizon and 7/8 is outside. We have no physical understanding of this curious distribution.

Note also that for closed spaces, such as an  $S^3$  type cosmology, the spatial hypersurface has no boundary, so it should have vanishing total energy. Hence the Witten positive energy proof (or indeed any other positivity proof) should not go through. We are not yet sure which step in the Witten argument breaks down for such closed spaces.

The Witten spinor Hamiltonian is also important for the quasilocal values it can yield. When integrated over a finite spatial region, the Hamiltonian boundary term  $\mathcal{B}_w$  (6.3) defines a quasilocal energy-momentum for any choice of the spinor field on the boundary. This is a popular approach to quasilocal energy. Several similar quasilocal boundary expressions of this type have been investigated. In this case one wants to determine the spinor field quasilocally (i.e., it should depend on the fields only on the boundary 2-surface  $S$ ). In particular Dougan and Mason [12] take  $\psi$  to be “holomorphic” (satisfying the 2-dimensional elliptic equation  $\gamma^A \nabla_A \psi = 0$ ) on the boundary. A nice investigation of the various options for equations to select the value of the spinor field for such quasilocal expressions has been carried out by Szabados [31].



## 6.2 An SU(2) formulation

Here we briefly describe a 3-dimensional alternative, similar to but distinct from the Witten formulation, which uses the SU(2) spinors of the 3-dimensional spatial hypersurface [26]. We begin from the well known ADM Hamiltonian [22, 18]

$$H(N) = \int d^3x \left\{ N \left[ g^{-\frac{1}{2}} (\pi^{mn} \pi_{mn} - \frac{1}{2} \pi^2) - g^{\frac{1}{2}} R \right] + 2\pi^m{}_k \nabla_m N^k \right\} + \oint dS_k N \delta_{am}^{kc} g^{mb} \Gamma^a{}_{bc}, \quad (6.8)$$

which we have supplemented by boundary term expressions that are valid in asymptotic Cartesian frames. In this case we concern ourself only with energy (not momentum), so we take the shift  $N^k$  to vanish. Within the ADM Hamiltonian density the scalar curvature term and the boundary term do not have a definite sign; the idea is to replace them with alternatives that are more definite. The 3-scalar curvature can be replaced using  $N = \varphi^\dagger \varphi$  and the 3-dimensional SCI

$$2 \left[ \nabla(\varphi^\dagger i\sigma) \wedge \nabla\varphi - \nabla\varphi^\dagger \wedge \nabla(i\sigma\varphi) \right] \equiv dB_{su2} - \varphi^\dagger \varphi R^{ab} \wedge \epsilon_{abc} \vartheta^c, \quad (6.9)$$

where  $R^{ab} \wedge \epsilon_{abc} \vartheta^c = R g^{1/2} d^3x$ ,  $\sigma := \sigma_a \vartheta^a$  is a Pauli matrix valued one form, and

$$B_{su2} := 2[\nabla\varphi^\dagger \wedge i\sigma\varphi + \varphi^\dagger i\sigma \wedge \nabla\varphi], \quad (6.10)$$

is a legitimate (since they agree asymptotically) alternative to the boundary term in (6.8).

The Hamiltonian density now takes the form

$$\mathcal{H}(\varphi) = (\varphi^\dagger \varphi) g^{-\frac{1}{2}} \left[ \pi^{mn} \pi_{mn} - \frac{1}{2} \pi^2 \right] d^3x + \left[ \nabla(\varphi^\dagger i\sigma) \wedge \nabla\varphi - \nabla\varphi^\dagger \wedge \nabla(i\sigma\varphi) \right]. \quad (6.11)$$

By arguments similar to those used in connection with the Witten Hamiltonian, the quadratic  $\nabla\varphi$  terms can be diagonalized to the form  $|\nabla_k \varphi|^2 - |\sigma^k \nabla_k \varphi|^2$ . Consequently the Hamiltonian density is non-negative—and hence the total energy is positive—on *maximal* spacelike hypersurfaces (in this standard time gauge condition the trace of the ADM canonical momentum  $\pi$  vanishes) if the 3-dimensional spinor is chosen to satisfy the 3-dimensional elliptic equation  $\sigma^k \nabla_k \varphi = 0$ .

Again, the boundary term  $B_{su2}$  gives a quasilocal energy for any choice of  $\varphi$ ; in particular, as in the previous case, we can use holomorphic spinors satisfying  $\sigma^A \nabla_A \varphi = 0$ .

### 6.3 The QSL approach

Our last alternative has some similar features to the above two approaches but differs from them in a very fundamental way. In the above the spinor field was introduced into the Hamiltonian as a technical device to aid in obtaining a locally non-negative Hamiltonian density. Instead one can introduce a spinor field into the Lagrangian, then it becomes a basic dynamical gravitational field. The key is another *spinor-curvature* identity, this time involving the 4-dimensional scalar curvature, which led us to a *Quadratic Spinor Lagrangian* (QSL) for GR [28]. (For the relation to teleparallel GR, aka TEGR,  $\text{GR}_{||}$ , see [32]) The Einstein-Hilbert scalar curvature Lagrangian equals (up to an exact differential) the QSL

$$\mathcal{L}_{qs} := 2D\bar{\Psi}\gamma \wedge D\Psi \equiv -R * 1 + d(D\bar{\Psi} \wedge \gamma\Psi + \bar{\Psi}\gamma \wedge D\Psi), \quad (6.12)$$

where  $\Psi = \vartheta\psi$  is a spinor one-form field. The spinor field is to be varied subject to the normalization constraints  $\bar{\psi}\psi = 1$ ,  $\bar{\psi}\gamma\psi = 0$  (which can be enforced via Lagrange multipliers).

In order to construct the Hamiltonian it is convenient to work from the corresponding first order Lagrangian:

$$\mathcal{L}_\Psi := D\bar{\Psi} \wedge P + \bar{P} \wedge D\Psi + \frac{1}{2}\bar{P} \wedge \gamma P \quad (6.13)$$

(which yields the pair of first order equations:  $2D\Psi = -\gamma P$  and its conjugate). From this we find the covariant Hamiltonian 3-form

$$\begin{aligned} \mathcal{H}_\Psi &:= \bar{P} \wedge \mathcal{L}_N \Psi + \mathcal{L}_N \bar{\Psi} \wedge P - i_N \mathcal{L}_\Psi \\ &\equiv -i_N \left( \frac{1}{2} \bar{P} \wedge \gamma P \right) \\ &\quad - \left[ i_N \bar{\Psi} DP + D\bar{\Psi} \wedge i_N P + \bar{\Psi} \wedge i_N \omega P - d(i_N \bar{\Psi} P) + \text{c.c.} \right], \end{aligned} \quad (6.14)$$

(the time derivative here is given by the Lie derivative). The Hamiltonian boundary term

$$\mathcal{B}(N) = i_N \bar{\Psi} P + \bar{P} i_N \Psi, \quad (6.15)$$

again yields quasilocal values for any choice of spinor field on the boundary.

One distinguishing feature of this approach, in which the spinor field is introduced into the Lagrangian, is that the displacement vector field  $N$  remains entirely independent of the spinor field. Another is that the connection does not appear as a primary dynamic variable. Nevertheless it is again possible to arrange that the Hamiltonian is locally non-negative. By arguments similar to those used before, with the lapse-shift choice  $N^\mu = (f^2, 0, 0, 0)$ , and the spinor field satisfying the (3-dimensional, elliptic) conformal Witten equation:  $\gamma^k D_k(f\psi) = 0$ , the QSL Hamiltonian density is locally non-negative on maximal slices. This yields another locally positive ‘localization’ along with a positive energy proof.

For further discussion of additional details and features of these cases see the already cited references as well as the overview in [5].

## 7 Properties of the spinor expressions

The spinor quasilocal expressions have certain common properties. A noteworthy one is that the expressions are essentially algebraic in  $N^\mu$ ; unlike the standard GR expression (5.3) they have no  $DN$  terms. (The importance of this feature will be discussed soon.) Another is perhaps the real *beauty* of these spinor formulations: they do not need an *explicit* reference configuration—whereas in the standard GR expression (5.3) the reference configuration is essential. When examined in detail it can be seen that the spinor field *implicitly* determines the reference values [8, 5]. That is one of its jobs.

There are other distinct roles for the spinor fields. The *Hamiltonian variation boundary principle* tells us what must be held fixed on the boundary. This clarifies the role of the fields. Essentially for the spinor formulations we find that (in addition to fixing the usual frame components) the spinor field  $\psi$  should be held fixed on the boundary. In each of the cases this has a distinct significance. For the Witten Hamiltonian fixing the spinor field fixes the displacement  $N^\mu$ , for the  $SU(2)$  spinor alternative it means holding the *lapse* fixed, whereas for the QSL fixed  $\psi$  amounts to fixing the observer's orthonormal frame on the boundary. In each case the spinor field has a different role. Altogether we have good examples of what spinors can mean geometrically and physically.

In each case, for displacements corresponding to 4-dimensional spacetime translations, the quasilocal spinor expressions (6.3,6.10,6.15) asymptotically agree with a standard one (5.3) and hence each expression gives similar reasonable values for the total energy-momentum.

## 8 Angular Momentum and Center-of-Mass Moment

However energy-momentum is not the whole story. For an asymptotically flat space each asymptotic symmetry has its associated conserved quantities. Energy-momentum is associated with spacetime translations. Such spaces also have asymptotic rotation and Lorentz boost symmetry. A proper physical Hamiltonian formalism allows for displacements which have the asymptotic Poincaré form

$$N^\mu = N_\infty^\mu + \lambda^\mu{}_\nu x^\nu. \quad (8.1)$$

Here  $N_\infty^\mu$  is a constant spacetime translation and the constants  $\lambda^{\mu\nu} = \lambda^{[\mu\nu]}$  describe an infinitesimal Lorentz transformation (including rotations and boosts). The value of the Hamiltonian generating the Lorentz displacements then gives 6 additional conserved quantities [29, 2]. The physical

significance of these additional quantities is most easily recognized by recalling that for a relativistic particle

$$L^{\mu\nu} := x^\mu p^\nu - x^\nu p^\mu \quad (8.2)$$

includes both *angular momentum*,

$$L^{ij} := x^i p^j - x^j p^i, \quad \text{and} \quad L^{0k} := x^0 p^k - x^k p^0, \quad (8.3)$$

the *center-of-mass moment*.

The conventional variable GR Hamiltonian description of section 5 does yield good values for these additional quantities for gravitating systems. An important contribution here is the  $i_N \Gamma \sim DN$  connection-Møller-Komar type term in (5.3) (such terms have often been overlooked in quasilocal investigations [3]). We found that this term plays a key role in: (i) black hole thermodynamics [6], (ii) certain angular momentum calculations [15, 16, 33] (iii) *all* center-of-mass moment calculations [20]. (Note: from (8.1) it follows that asymptotically the  $DN$  term has a contribution proportional to  $\lambda$ .)

Now such terms are completely absent in all three of our spinor formulations. Without them we, (i) do not know how to obtain the first law of black hole thermodynamics, (ii) have difficulties in obtaining the angular momentum (within the Witten Hamiltonian (6.2), modifying the displacement to  $\bar{\psi}\gamma^\mu\psi$  will work—but only *if* we take the strange asymptotics  $\psi^\dagger\psi \sim r$ ), (iii) simply *cannot see how* to get the correct center-of-mass moment. Now we have not entirely given up trying, but so far we have not managed to include the effects of such  $D^{[\mu}N^{\nu]}$  type terms in the spinor formulations. Presently we are preparing detailed discussions of these crucial issues. Here we can summarize our tentative conclusions but regretfully cannot include any more of the technical supporting details.

## 9 Conclusions

We considered the spinor formulations for the Hamiltonian and the associated positive energy proofs and quasilocal expressions. At first the role of the spinor field seemed mysterious. However the boundary variation principle clarifies the role of the spinor field (and indeed all other variables). Our spinor Hamiltonian expressions illustrate the variety of roles that spinor fields can play. Spinors give beautiful positive energy proofs (especially Witten's) and very neat formulas for quasilocal energy-momentum. It is especially noteworthy that there is no need for extra reference fields on the boundary.

Yet it seems that such spinor formulations have a serious limitation: the present expressions cannot give angular momentum and the center-of-mass

moment. Moreover, we do not see how any natural adjustment of these spinor formulations for quasilocal Hamiltonian boundary terms can successfully give expressions for these quantities. Apparently the spinor Hamiltonians cannot give all of the physically conserved quantities of asymptotically flat spacetimes. Hence, although they are deservedly popular and quite good for many purposes, we now believe that such spinor formulations cannot really capture *in a physically correct way* the still elusive gravitational energy-momentum.

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